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Problem sheet 6: Quantum magnetism **21.01.2025**

You can use both the SI or CGS units, but you may find CGS easier whenever magnetic moments and magnetic susceptibilities are involved.

6.1. Spin operators (6 P)

Use the standard commutation relations,

$$[\hat{S}_j^x, \hat{S}_j^y] = i\hbar\, \hat{S}_j^z, \qquad [\hat{S}_j^y, \hat{S}_j^z] = i\hbar\, \hat{S}_j^x, \qquad [\hat{S}_j^z, \hat{S}_j^x] = i\hbar\, \hat{S}_j^y$$

to compute:

(a) $[\hat{S}_j^+, \hat{S}_j^-]$

(b) $[\hat{S}_j^+, \hat{S}_j^z]$ and $[\hat{S}_j^-, \hat{S}_j^z]$

(c) $[\mathcal{H}, \hat{S}^z]$ and $[\mathcal{H}, \hat{\mathbf{S}}^2]$

where $\hat{S}^z = \hat{S}_1^z + \hat{S}_2^z$, $\hat{\mathbf{S}}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2$, and $\mathcal{H} = J\, \hat{\mathbf{S}}_1 \hat{\mathbf{S}}_2$.

6.2. Spin-1 dimer (10 P)

Solve the Heisenberg model for a spin dimer,

$$\mathcal{H} = J\, \hat{\mathbf{S}}_1 \hat{\mathbf{S}}_2$$

where $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ are spin-1 operators ($S_1 = S_2 = 1$).

(a) Construct the basis states $|S_1^z, S_2^z\rangle$ and distribute them into different groups according to their $\langle S^z \rangle$ value where $\hat{S}^z = \hat{S}_1^z + \hat{S}_2^z$.

(b) The matrix of $\mathcal{H}$ should be block-diagonal. Each of its sectors corresponds to a given $\langle S^z \rangle$ value. Write down each of these parts.

(c) Diagonalize them and obtain the full energy spectrum. Write the states explicitly and label them with their $\langle \hat{S}^z \rangle$ as well as $\langle \hat{\mathbf{S}}^2 \rangle$ values.

(d) Sketch the magnetization curve (M vs. B at $T \to 0$) and determine the critical fields.

6.3. Anisotropic spin dimer (4 P)

Consider a spin dimer with $S_1 = S_2 = \frac{1}{2}$ and the XXZ Hamiltonian,

$$\mathcal{H} = J_x\, \hat{S}_1^x \hat{S}_2^x + J_x\, \hat{S}_1^y \hat{S}_2^y + J_z\, \hat{S}_1^z \hat{S}_2^z$$

Repeat the steps shown in lecture 10 and calculate the energy spectrum. How is it different from the Heisenberg case ($J_x = J_z$)?