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Advanced Solid State Physics
WS 24/25, Uni Leipzig

Problem sheet 10: Collective spin states **22.01.2025**

10.1. Spin-$\frac{1}{2}$ triangle

Consider three spins forming a triangle. They are described by the Hamiltonian

$$\mathcal{H} = J(\hat{\mathbf{S}}_1\hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_1\hat{\mathbf{S}}_3 + \hat{\mathbf{S}}_2\hat{\mathbf{S}}_3)$$

where $S_1 = S_2 = S_3 = \frac{1}{2}$ and $J > 0$. Solve this Hamiltonian similar to the spin-$\frac{1}{2}$ dimer shown in lecture 21.

(a) Construct the basis set. How many states does it contain, and how many eigenstates of $\mathcal{H}$ do you expect?

(b) Show that $\mathcal{H}$ commutes with $\hat{S}^z = \hat{S}_1^z + \hat{S}_2^z + \hat{S}_3^z$. You can then use S^z as a quantum number to distribute your basis states into several groups.

(c) Write down the block-diagonal matrix of $\mathcal{H}$.

(d) Diagonalize the matrix and obtain the eigenstates. Label them according to their S^z and S^2 values where $\hat{\mathbf{S}}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_3)^2$.

(e) Compare the result to the spin-$\frac{1}{2}$ dimer shown in the lecture. Why is the ground state non-degenerate in the case of the dimer and degenerate in the case of the triangle?

10.2. Classical version of spin waves

Obtain spin-wave excitations without quantum mechanics. Consider a chain of spins S with the exchange coupling J.

(a) Use the energy from the classical version of the Heisenberg model,

$$E = J\sum_p \mathbf{S}_p\mathbf{S}_{p+1}$$

to obtain the energy associated with spin p,

$$E_p = -g\mu_B\mathbf{S}_p\mathbf{B}_p^{\text{eff}}$$

where $g = 2$ and $\mathbf{B}_p^{\text{eff}}$ is the effective magnetic field acting on $\mathbf{S}_p$.

(b) Use the equation of motion,

$$h\frac{d\mathbf{S}_p}{dt} = g\mu_B\mathbf{S}_p \times \mathbf{B}_p^{\text{eff}},$$

to derive the spin-wave excitation. To this end, assume that S_p^x and S_p^y are small, whereas $S_p^z \simeq S$ (the excitation is a small tilt of the spins away from the z-axis). Write the equations of motion for S_p^x and S_p^y assuming the ferromagnetic ground state of the spin chain ($J < 0$).

(c) Use the ansatz

$$S_p^x = u\,e^{i(pqa-\omega t)}, \qquad S_p^y = v\,e^{i(pqa-\omega t)}$$

to solve the equations from (b). Here, u and v are the amplitudes, q is the wave number (1D propagation vector), and a is the lattice parameter. Show that $u = -iv$ (hence the $\pi/2$ shift between the oscillations of the x- and y-spin components) and obtain the dispersion relation for the ferromagnetic chain.

(d) Repeat the steps (b) and (c) for an antiferromagnet ($J > 0$). In this case, $S_p^z = (-1)^p S$, and you should write four equations of motion, two for S_p^x and S_p^y and two more for S_{p+1}^x and S_{p+1}^y, respectively. Combine them into two equations for S_p^+ and S_{p+1}^+ (the $\pi/2$ shift between S_p^x and S_p^y is now implicitly taken into account within $S_p^+ = S_p^x + iS_p^y$).

Submit the written solution to Problem 10.1 in the class on 5.02.25 at 15:15
Tutorial session on 5.02.25 at 15:15

(e) Now solve these equations using the ansatz

$$S_p^+ = u\, e^{i(pqa-\omega t)}, \qquad S_{p+1}^y = v\, e^{i((p+1)qa-\omega t)}$$

Obtain the dispersion relation for an antiferromagnet and compare it to the ferromagnetic case.