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Problem sheet 4: Chern insulators **13.11.2024**

4.1. Spin-$\frac{1}{2}$ Hamiltonian and its application to honeycomb crystals

Consider the spin-$\frac{1}{2}$ Hamiltonian, $\mathcal{H} = \mathbf{B}\,\boldsymbol{\sigma}$ where $\mathbf{B}$ is the magnetic field and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.

(a) Show that the eigenstates of $\mathcal{H}$ are given by

$$|u_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \\ \sin\frac{\theta}{2}\,e^{i\varphi} \end{pmatrix}, \qquad |u_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \\ -\cos\frac{\theta}{2}\,e^{i\varphi} \end{pmatrix}$$

where (B, θ, φ) are the spherical coordinates that define the length and direction of $\mathbf{B}$.

(b) Calculate the Berry connection $\mathbf{A}$ for both eigenstates. Use spherical coordinates.

(c) Calculate the Berry curvature $\boldsymbol{\Omega}$ for both eigenstates. Keep using the spherical coordinates.

(d) Now apply the same formalism to the Hamiltonian of 2D massive Dirac fermions,

$$\mathcal{H} = v(\sigma_x\hat{p}_x + \sigma_y\hat{p}_y) + m\sigma_z.$$

Show that the Chern number (integral of $\boldsymbol{\Omega}$ over the $p_x - p_y$ momentum space) is given by

$$C_{\text{Dirac}} = \frac{1}{2\pi}\iint dp_x\,dp_y\,\Omega(\mathbf{p}) = -\frac{1}{2}\,\text{sign}(m).$$

This calculation will basically elucidate the Chern numbers in the Haldane model (lecture 8).

4.2. Berry curvature in a 2D material

Stanene (tinene) is a hexagonal sheet of tin atoms, a somewhat fictitious analog of graphene. Use calculated band structure of this material to determine:

(a) electron velocity in the vicinity of K

(b) Berry curvature at the point, which is $0.2\,\text{Å}^{-1}$ away from K

You will find the band structure in Fig. 2d of J. Phys.: Condens. Matter 25, 395305 (2013) where it is shown with the red lines. Take the lattice parameter a from the same publication. You will also see their estimate of the Fermi velocity that can be compared with your result.

Bonus question: why does the gap at K increase upon going from graphene to stanene?

4.3. Chiral edge modes of Chern insulators

Consider the following Dirac Hamiltonian with a spatially dependent mass,

$$\mathcal{H} = v\,(\hat{p}_x\boldsymbol{\sigma}_x + \hat{p}_y\boldsymbol{\sigma}_y) + m(x)\,\boldsymbol{\sigma}_z$$

where

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \qquad \hat{p}_y = -i\hbar\frac{\partial}{\partial y}, \qquad m(x) = m\,\text{sign}(x) = \begin{cases} m, & x > 0 \\ -m, & x < 0 \end{cases}$$

Choose $v > 0$ but allow mass to be either positive or negative.

Submit the written solution to Problem 4.3 in the class on 27.11.24 at 15:15
Tutorial session on 27.11.24 at 15:15

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Consider an ansatz of wavefunctions in the form

$$\psi(x,y) = e^{ik_y y}\, e^{-|x|/\xi} \begin{pmatrix} a \\ b \end{pmatrix}$$

where a and b are independent of both x and y.

(a) What sign should the localization length ξ have in order for the above ansatz to be exponentially decaying away from $x = 0$ and, therefore, to be a physically allowed solution?

(b) Determine the value of the localization length, ξ, the dispersion relation $E(k_y)$, and the vector (a, b) for the physically allowed solutions that are exponentially decaying away from $x = 0$.

(c) For which sign of the mass, m, do these waves move always toward the positive y-direction?