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Problem sheet 7: Second quantization. Landau Fermi liquid **18.12.2024**

7.1. Second quantization gymnastics

(a) Consider a set of boson operators $\hat{b}_a$ and $\hat{b}_a^\dagger$ $(a = 1, \ldots N)$ satisfying the usual canonical boson commutation relations. Then consider the set of boson bilinears of the form $\hat{b}_a^\dagger \hat{b}_b$.

Compute the following commutator of boson bilinears:

$$[\hat{b}_a^\dagger \hat{b}_b,\, \hat{b}_c^\dagger \hat{b}_d] = ?$$

(b) Repeat the above exercise for fermions, namely consider a set $\hat{c}_a$ and $\hat{c}_a^\dagger$ $(a = 1, \ldots N)$ of ordinary complex fermion operators satisfying the usual canonical anti-commutation relations. Then consider the set of fermion bilinears of the form $\hat{c}_a^\dagger \hat{c}_b$.

Compute the following commutator of fermion bilinears:

$$[\hat{c}_a^\dagger \hat{c}_b,\, \hat{c}_c^\dagger \hat{c}_d] = ?$$

Are the algebraic relations of commutators of bilinears of fermions **different** from those of bilinears of bosons?

7.2. Kadowaki-Woods ratio

Download the heat capacity and resistivity data for Sr_2RuO_4.

(a) Determine the Sommerfeld coefficient and the effective mass. The DFT result for the density of states at the Fermi level is $N(\varepsilon_F) = 4.36\,\mathrm{eV}^{-1}/\mathrm{f.u.}$

(b) Determine the A value in the Fermi-liquid expression for the resistivity, $\rho = \rho_0 + AT^2$.

(c) Calculate the Kadowaki-Woods ratio A/γ^2 and locate Sr_2RuO_4 on the A vs. γ^2 plot (you can find one in Phys. Rev. Lett. 94, 057201 (2005)).

7.3. Particle-hole continuum

Consider fermions in 2D with a parabolic dispersion, so that the Hamiltonian is

$$\mathcal{H} = \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2m} - \mu \right) \hat{c}_{\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}}.$$

The many-body eigenstates of the above Hamiltonian can be labeled by the occupation of each momentum,

$$\hat{c}_{\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}} |\Psi\rangle = n_{\mathbf{p}} |\Psi\rangle, \qquad n_{\mathbf{p}} \in \{0, 1\}.$$

The total energy and total momentum of the many-body eigenstate are then given by

$$\mathcal{H}|\Psi\rangle = E|\Psi\rangle, \qquad E = \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2m} - \mu \right) n_{\mathbf{p}}$$

and

$$\hat{\mathbf{P}}|\Psi\rangle = \mathbf{P}|\Psi\rangle, \qquad \mathbf{P} = \sum_{\mathbf{p}} \mathbf{p}\, n_{\mathbf{p}}.$$

The ground state has occupations given by $n_{\mathbf{p}}^0 = \Theta\left(\mu - \frac{\mathbf{p}^2}{2m}\right)$ where Θ is the Heaviside function.

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(a) Consider now an excited state containing one particle at momentum $\mathbf{k}+\mathbf{q}$ and one hole at momentum $\mathbf{k}$. What are the conditions that $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ must satisfy in order that such excited state is physically allowed? (keep in mind the Pauli-exclusion principle). **Compute** the energy of this particle-hole pair excitation

$$\varepsilon_{ph}(\mathbf{k},\mathbf{q}) = E_{ph}(\mathbf{k},\mathbf{q}) - E_0$$

where $E_{ph}(\mathbf{k},\mathbf{q})$ is the energy of the many-body state containing the particle and the hole, and E_0 is the ground-state energy. Also **compute** the total momentum relative to the ground state,

$$\mathbf{p}_{ph}(\mathbf{k},\mathbf{q}) = \mathbf{P}_{ph}(\mathbf{k},\mathbf{q}) - \mathbf{P}_0.$$

(b) Consider a fixed value of $\mathbf{q}$, which can be taken along the x-axis without loss of generality, namely, $\mathbf{q} = q\,\hat{\mathbf{x}}$ with $q > 0$. Now for each value of q, **find** the maximum and the minimum energy that the particle-hole pair can have when we allow $\mathbf{k}$ to change among the physically allowed values from part (a), namely, find the following,

$$\varepsilon_+(q) = \max_{\mathbf{k}} \varepsilon_{ph}(\mathbf{k}, q\,\hat{\mathbf{x}}), \qquad \varepsilon_-(q) = \min_{\mathbf{k}} \varepsilon_{ph}(\mathbf{k}, q\,\hat{\mathbf{x}}).$$

(c) **Make a plot** of the region of allowed energies of particle-hole pairs as a function of their momentum q, namely, plot $\varepsilon_-(q)$ and $\varepsilon_+(q)$. What is the slope of $\varepsilon_+(q)$ in the limit $q \to 0$, and what is the largest value of q that allows for $\varepsilon_-(q)$ to be zero?